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Mixed Lagrangian–Eulerian description of vortical flows for ideal and viscous fluids

E. A. KUZNETSOV

P. N. Lebedev Physical Institute, 53 Leninsky Ave., 119991 Moscow, Russia
and L. D. Landau Institute for Theoretical Physics, 2 Kosygin Str., 119334 Moscow, Russia

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It is shown that the Euler hydrodynamics for vortical flows of an ideal fluid is equivalent to the equations of motion of a charged *compressible* fluid moving due to a self-consistent electromagnetic field. The velocity of new auxiliary fluid coincides with the velocity component normal to the vorticity line for the primitive equations. Therefore this new hydrodynamics represents hydrodynamics of vortex lines. Their compressibility reveals a new mechanism for three-dimensional incompressible vortical flows connected with breaking (or overturning) of vortex lines which can be considered as one of the variants of collapses. Transition to the Lagrangian description in the new hydrodynamics corresponds, for the original Euler equations, to a mixed Lagrangian–Eulerian description – the vortex line representation (VLR). The Jacobian of this mapping defines the density of vortex lines. It is shown also that application of VLR to the Navier–Stokes equations results in an equation of diffusive type for the Cauchy invariant. The diffusion tensor for this equation is defined by the VLR metric.

1. Introduction

Collapse, as a process of singularity formation in a finite time from an initially smooth distribution, plays a very important role, being considered as one of the most effective mechanisms of energy dissipation. Collapse is also essential in considering hydrodynamics of incompressible fluids. It is well-known that the appearance of a singularity in gasdynamics, i.e. in compressible hydrodynamics, is connected with the phenomenon of breaking, which is the physical mechanism leading to the emergence of shocks. From the point of view of the classical catastrophe theory (Arnold 1986), this process is simply the formation of folds. It is completely characterized by the mapping which corresponds to the transition from the Eulerian description to the Lagrangian one. The vanishing of the Jacobian J for this mapping denotes an intersection of trajectories of the Lagrangian particles and the emergence of a singularity for spatial derivatives of the velocity and gas density. In the incompressible case, it seems that breaking does not exist because the Jacobian of the corresponding mapping is constant (in the simplest case equal to unity), and therefore there is no reason for the existence of such a phenomenon. Nevertheless, breaking is possible in this case as well (Kuznetsov & Ruban 2000*a*; Zheligovsky, Kuznetsov & Podvigina 2001; Kuznetsov, Podvigina & Zheligovsky 2003). It can also happen with vortex lines. A bundle of continuously distributed vortex lines can be compressed despite the incompressibility of the vorticity as a divergence-free field. The main reason for the compressibility of the vortex bundle is connected with the following observation. The three-dimensional

Euler equations for the vorticity $\boldsymbol{\Omega} = \operatorname{curl} \boldsymbol{v}$ for incompressible fluids,

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \operatorname{curl}[\boldsymbol{v} \times \boldsymbol{\Omega}], \tag{1.1}$$

contain the vector product between the velocity $\boldsymbol{v}$ and vorticity $\boldsymbol{\Omega}$. This means that the vorticity changes only due to the velocity component $\boldsymbol{v}_n$ normal to the vortex line. Due to the vorticity being frozen into fluid (the Helmholtz theorem, see e.g. Lamb 1932), this velocity component represents the velocity of the vortex line. The velocity component $\boldsymbol{v}_\tau$ parallel to the vorticity field plays a passive role, providing the incompressibility of the flow: $\nabla \cdot \boldsymbol{v}_n + \nabla \cdot \boldsymbol{v}_\tau = 0$. Therefore, the divergence of the normal velocity, in a general situation, is not equal zero. This, in fact, introduces a compressible entity into incompressible vortical flows that simultaneously opens the possibility for the breaking of vortex lines. Unlike the breaking in gasdynamics, the breaking of vortex lines means that one vortex line can reach another vortex line. For smooth initial conditions, breaking can happen for the first time when the vortex lines touch each other at a single point. At the touching point the vorticity becomes infinite.

To describe the breaking of vortex lines, in Kuznetsov & Ruban (1998, 2000*b*) we introduced the so-called vortex line representation (VLR), namely a mixed Lagrangian–Eulerian description, where each vortex line is labelled by a two-dimensional marker while another parameter defines the vortex line itself. The necessity of introducing VLR is connected with the existence of a hidden symmetry in the Euler equations, usually referred to as the relabelling symmetry (found by Salmon 1988). The famous Kelvin theorem concerning the conservation of velocity circulation as well as the Cauchy invariants, less known to a wide audience of fluid dynamicists, are direct consequences of this symmetry. The Cauchy invariants, being Lagrangian invariants, are given at each Lagrangian point and, as a consequence, at each point of the physical space. For two-dimensional flows these invariants are simply the vorticity, advected by the fluid. In the three-dimensional geometry these invariants cannot be expressed locally in terms of such characteristic properties as velocity, vorticity, or pressure. Thus, the dynamics of ideal fluids is very restricted due to the presence of an infinite (continuous) number of constraints. The vortex line representation resolves this problem. The application of the VLR to the Euler equation results in partially integrated equations resolved explicitly with respect to this infinite number of integrals. This is very important for the numerical solution of the Euler equations, especially if it concerns numerical studies of collapse processes.

In this paper, we review the further development of the VLR for both ideal and viscous fluids, given in Kuznetsov (2002), focusing on a detailed derivation of VLR application to the Navier–Stokes equations for incompressible fluids and also a modification of this method to the case when the vorticity field contains zero points. The latter is important for vortex reconnection which, in accordance with the numerical experiments by Kerr (1993, 2005), is considered one of the main processes leading to collapse. We demonstrate also how the VLR is modified for both pure a Lagrangian description with new trajectories defined by vortex lines and a pure Eulerian description.

The paper is organized as follows. In the next section, we review some basic properties of the Euler hydrodynamics such as the Kelvin theorem, the Cauchy invariants, and the Weber transform. It is shown that both conservation laws, i.e. the Kelvin theorem on the conservation of velocity circulation and the Cauchy invariants, represent the same phenomenon. The difference between them is that the Kelvin theorem concerns the conservation of the *integral* quantity, i.e. the velocity circulation, while the Cauchy invariant is *local*, expressing the same constancy. Both conservation laws can also be considered as consequences of the ‘frozenness’ of the vorticity into the fluid. According to this property, fluid particles are attached to their vortex line and cannot leave it. In this section we also show that the so-called Kuzmin decomposition (Kuzmin 1983) represents the Weber transform.

In § 3, we clarify the role of the Clebsch variables in the vortex line representation: these variables can be used as Lagrangian markers of the vortex lines. However, as is well-known, these variables can be introduced locally and, generally speaking, cannot be extended to the entire space because flows parameterized by the Clebsch variables have a vanishing helicity integral $\int(\boldsymbol{v}\cdot\mathrm{curl}\,\boldsymbol{v})\mathrm{d}\boldsymbol{r}$, which is a topological invariant characterizing the degree of knottiness of the vortex lines (Moffatt 1969).

In § 4 we show that, in a general situation, the Euler equations can be rewritten as the equations of motion for a charged *compressible* fluid moving under the action of effective self-consistent electric and magnetic fields satisfying Maxwell’s equations (Kuznetsov 2002). The new velocity coincides with the velocity component transverse to the vorticity, which, due to its ‘frozenness’ property, is identified with the vortex line velocity. This new hydrodynamics can be called the hydrodynamics of vortex lines. The most important property of this hydrodynamics is the compressibility of the auxiliary new fluid. This implies the compressibility of the mapping corresponding to the transition from the Eulerian to the Lagrangian description and, respectively, a possibility of breaking. In terms of the Eulerian properties, this results in the breaking of vortex lines when the vorticity becomes infinite. In the framework of the new hydrodynamics of a charged fluid, the role of the density is played by the quantity inverse to J. This hydrodynamic variable is naturally called the density of the vortex lines. J^{-1} appears from the Cauchy formula for the vorticity $\boldsymbol{\Omega}$. The evolution of the vortex line density in time and space is defined by the velocity component normal to the vorticity. As it is shown in this paper, the Cauchy formula can be obtained from a ‘new’ Kelvin theorem as well as from the analogue of the Weber transformation. As a result, the Euler equations are resolved with respect to the Cauchy invariants, i.e. relative to the infinite number of integrals of motion. In this section we also discuss how the VLR, as a local change of variables, can be modified when the vorticity is equal to zero and hence the normal velocity is uncertain. Such null points are considered as topological singularities with integer topological charge, which is constant in time.

Section 5 is devoted to an application of the VLR to viscous incompressible fluids described by the Navier–Stokes equations when the Cauchy invariant becomes time dependent and obeys a diffusion-type equation with a ‘diffusion tensor’ determined by the VLR metric and viscosity coefficient ν. In this case the equations of motion of vortex lines in their original (for ideal fluids) form can be considered as a transformation to a new curvilinear system of coordinates. The exact equations obtained for viscous flows can be interpreted as an exact separation of two different temporal scales: the inertial (in fact, the nonlinear) and the viscous scales. The discussion and summary are given in § 6.

2. General remarks

The Euler equations for an ideal incompressible fluid (with density $\rho = 1$),

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p, \qquad \nabla\cdot\boldsymbol{v} = 0, \tag{2.1}$$

in both two and three dimensions possess an infinite (continuous) number of integrals of motion (see e.g.; Lamb 1932; Salmon 1988; Zakharov & Kuznetsov 1997). These integrals are called the Cauchy invariants. The simplest way to derive the Cauchy invariants is to use the Kelvin theorem on the conservation of velocity circulation,

$$\Gamma = \oint (\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l}), \tag{2.2}$$

where the integration contour $C[\boldsymbol{r}(t)]$ moves together with the fluid. Transforming from the Eulerian $\boldsymbol{r}$ to the Lagrangian coordinates $\boldsymbol{a}$ allows us to rewrite (2.2) as

$$\Gamma = \oint \dot{x}_i \frac{\partial x_i}{\partial a_k} \, \mathrm{d}a_k,$$

where the new contour $C[\boldsymbol{a}]$ is immovable and the dot represents the time derivative for fixed $\boldsymbol{a}$. (Here and everywhere below repeated indices imply summation.) Because the contour $C[\boldsymbol{a}]$ is arbitrary, and using the Stokes formula, one can conclude that the quantity

$$\boldsymbol{I}(\boldsymbol{a}) = \mathrm{curl}_a \left(\dot{x}_i \frac{\partial x_i}{\partial \boldsymbol{a}} \right) \tag{2.3}$$

is preserved in time for each point $\boldsymbol{a}$. This is simply a Cauchy invariant. If the Lagrangian coordinates $\boldsymbol{a}$ in (2.3) coincide with the initial positions of the fluid particles, the invariant $\boldsymbol{I}$ is equal to the initial vorticity $\boldsymbol{\Omega}_0(\boldsymbol{a})$. Thus, the Cauchy invariants, being Lagrangian invariants, are given at each Lagrangian point $\boldsymbol{a}$ and, as a consequence, at each point of the physical space. This makes the dynamics of ideal fluids very restricted due to the presence of an infinite (continuous) number of constraints. For two-dimensional flows, the vector $\boldsymbol{I}(\boldsymbol{a})$ has only one component perpendicular to the flow plane, which yields the well-known relation $\Omega = I(\boldsymbol{a})$, i.e. the vorticity, being a Lagrangian invariant, is advected by the fluid. In the three-dimensional case, however, the vector $\boldsymbol{I}(\boldsymbol{a})$ cannot be expressed locally through the velocity $\boldsymbol{v}$, the vorticity $\boldsymbol{\Omega}$ or the pressure p.

The Cauchy invariants can be also derived from the Weber transformation (see e.g. Lamb 1932; Zakharov & Kuznetsov 1997 and references therein). This transformation is constructed from the Lagrangian form of the Euler equations

$$\frac{\partial x_i}{\partial a_k} \ddot{x}_i = -\frac{\partial p}{\partial a_k} \tag{2.4}$$

where the Jacobi matrix $J_{ik} = \partial x_k / \partial a_i$ appears on the left-hand side of the equation after excluding the $\boldsymbol{r}$-dependence for ∇p. Upon introducing the vector

$$u_k = v_i \frac{\partial x_i}{\partial a_k},$$

which depends on t and $\boldsymbol{a}$, equation (2.4) can be rewritten in the form

$$\dot{\boldsymbol{u}} = -\nabla_a \left(p - \frac{v^2}{2} \right). \tag{2.5}$$

Integrating this equation with respect to time results in the transform suggested by Weber in 1868 (see (Lamb 1932))

$$\boldsymbol{u}(\boldsymbol{a}, t) = \boldsymbol{u}_0(\boldsymbol{a}) + \nabla_a \Phi, \tag{2.6}$$

where the potential Φ is defined from the non-stationary Bernoulli equation,

$$\dot{\Phi} = -p + \frac{v^2}{2},$$

with $\Phi|_{t=0} = 0$. In this case the time-independent vector $\boldsymbol{u}_0(\boldsymbol{a})$ coincides with the initial velocity $\boldsymbol{v}_0(\boldsymbol{a})$. In the physical $\boldsymbol{r}$-space the Weber formula (2.6) becomes

$$\boldsymbol{v}(\boldsymbol{r}, t) = v_{0k}(\boldsymbol{a})\nabla a_k + \nabla\Phi. \tag{2.7}$$

One can easily verify that the vector $\boldsymbol{w} = v_{0k}(\boldsymbol{a})\nabla a_k$ in (2.7) is a solution to the equation

$$\frac{\partial \boldsymbol{w}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{w} = -(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}.$$

This equation was derived in Kuzmin (1983), recently in comparison with 1868 when Weber found his transform. In fact, Kuzmin rediscovered the Weber decomposition; in the literature the decomposition (2.7) is sometimes called the Kuzmin decomposition.

Applying the curl operator to the relation (2.6) leads again to the Cauchy invariant (2.3). Conservation of the Cauchy invariants, as was shown for the first time by Salmon (1988) (see also Zakharov & Kuznetsov 1997), is a consequence of the special (infinite) symmetry, the so-called relabelling symmetry. These invariants characterize the 'frozenness' of the vorticity into the fluid. This is a very important property, and means that the fluid (Lagrangian) particles cannot leave their own vortex lines, i.e. the vortex lines on which they were initially. Thus, the Lagrangian particles have one independent degree of freedom corresponding to the motion along the vortex lines. On the other hand, it follows from the equation for the vorticity (1.1) that such a motion does not change the value of the vorticity. From this point of view a vortex line represents an invariant object and therefore it is natural to seek a transformation for which this invariance is clearly seen from the beginning. Such a description – the vortex line representation – was introduced in Kuznetsov & Ruban (1998, 2000*b*).

3. Mixed Langragian–Eulerian description

Consider a vortical flow ($\boldsymbol{\Omega} \neq 0$) of an ideal fluid which can be described by the Clebsch variables λ and μ (see e.g. Lamb 1932; Kuznetsov & Mikhailov 1980):

$$\boldsymbol{\Omega} = [\nabla\lambda \times \nabla\mu]. \tag{3.1}$$

The geometrical meaning of these variables is well known: an intersection of two surfaces $\lambda = \text{const}$ and $\mu = \text{const}$ yields the vortex line. It is also known that in the incompressible case the Clebsch variables are Lagrangian invariants, i.e. they are unchanged along the trajectories of the fluid particles:

$$\frac{\partial\lambda}{\partial t} + (\boldsymbol{v}\cdot\nabla)\lambda = 0 \qquad \frac{\partial\mu}{\partial t} + (\boldsymbol{v}\cdot\nabla)\mu = 0. \tag{3.2}$$

Therefore, these variables can be taken as markers for the vortex lines. It is easy to establish that the transition in (3.1) to the new variables

$$\lambda = \lambda(x, y, z), \qquad \mu = \mu(x, y, z), \qquad s = s(x, y, z), \tag{3.3}$$

where the vortex line is labelled by λ and μ and parametrized by s, leads to

$$\boldsymbol{\Omega}(\boldsymbol{r}, t) = \frac{1}{J}\cdot\frac{\partial \boldsymbol{r}}{\partial s}. \tag{3.4}$$

Here

$$J = \frac{\partial(x, y, z)}{\partial(\lambda, \mu, s)}$$

is the Jacobian of the mapping

$$\boldsymbol{r} = \boldsymbol{r}(\lambda, \mu, s). \tag{3.5}$$

The transform (3.5) is inverse to (3.3); it defines the corresponding transition to the curvilinear coordinate system connected to the movable vortex lines. In this case λ and μ are Lagrangian variables, but the parameter s has the meaning of an Eulerian coordinate.

The equations of motion of the vortex lines (the equations for $\boldsymbol{r}(\lambda, \mu, s, t)$) can be obtained directly from (1.1) for the motion for the vorticity. The simplest way to derive these is to use a linear combination of equations (3.2),

$$\nabla\mu \left[\frac{\partial\lambda}{\partial t} + (\boldsymbol{v}\cdot\nabla)\lambda\right] - \nabla\lambda \left[\frac{\partial\mu}{\partial t} + (\boldsymbol{v}\cdot\nabla)\mu\right] = 0, \tag{3.6}$$

which is equivalent to (3.2), due to the linear independence of the vectors $\nabla\lambda$ and $\nabla\mu$.

Performing the transformations (3.3) in (3.6), we arrive at the equation of motion for the vortex lines (Kuznetsov & Ruban 1998)

$$\left[\frac{\partial \boldsymbol{r}}{\partial s} \times \left(\frac{\partial \boldsymbol{r}}{\partial t} - \boldsymbol{v}(\boldsymbol{r}, t)\right)\right] = 0. \tag{3.7}$$

This equation has one important property: any motion along a vortex line does not change the line itself. It is easy to check that (3.7) is equivalent to

$$\frac{\partial \boldsymbol{r}}{\partial t} = \boldsymbol{v}_n(\boldsymbol{r}, t), \tag{3.8}$$

where $\boldsymbol{v}_n$ is the velocity component normal to the vorticity vector.

In accordance with the Darboux theorem, the Clebsch variables can always be introduced locally, but not globally. Flows parameterized by the Clebsch variables have a vanishing helicity integral $\int(\boldsymbol{v}\cdot\mathrm{curl}\,\boldsymbol{v})\mathrm{d}\boldsymbol{r}$, which is a topological invariant characterizing the degree of knottiness of the vortex lines (Moffatt 1969). Therefore, to introduce the vortex line representation for flows with non-trivial topology, it is necessary to return to the original equations of motion (2.1) and (1.1) for the velocity and vorticity.

4. Vortex line representation

According to equation (1.1), the velocity component $\boldsymbol{v}_\tau$ tangent to the vector $\boldsymbol{\Omega}$ does not affect (directly) the vorticity dynamics, i.e. in (1.1) we can put, instead of $\boldsymbol{v}$, its transverse component $\boldsymbol{v}_n$.

The equation of motion for the transverse velocity $\boldsymbol{v}_n$ follows directly from equation (2.1). It has the form of the equation of motion of a charged particle moving in an electromagnetic field:

$$\frac{\partial \boldsymbol{v}_n}{\partial t} + (\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n = \boldsymbol{E} + [\boldsymbol{v}_n \times \boldsymbol{H}], \tag{4.1}$$

where the effective electric and magnetic fields are given by

$$\boldsymbol{E} = -\nabla\left(p + \frac{v_\tau^2}{2}\right) - \frac{\partial \boldsymbol{v}_\tau}{\partial t}, \tag{4.2}$$

$$\boldsymbol{H} = \mathrm{curl}\,\boldsymbol{v}_\tau, \tag{4.3}$$

respectively. The electric and magnetic fields introduced above are expressed in terms of scalar and vector potentials, φ and $\boldsymbol{A}$ respectively, in the standard way:

$$\varphi = p + \frac{\boldsymbol{v}_\tau^2}{2}, \qquad \boldsymbol{A} = \boldsymbol{v}_\tau,$$

so that two Maxwell's equations,

$$\nabla \cdot \boldsymbol{H} = 0, \qquad \frac{\partial \boldsymbol{H}}{\partial t} = -\operatorname{curl} \boldsymbol{E},$$

are satisfied automatically. In this case the vector potential $\boldsymbol{A}$ has the gauge

$$\nabla \cdot \boldsymbol{A} = -\nabla \cdot \boldsymbol{v}_n,$$

which is equivalent to the condition $\nabla \cdot \boldsymbol{v} = 0$.

It follows from (4.2) and (4.3) that the two other Maxwell's equations can be considered as the definitions of the charge density ρ and the current $\boldsymbol{j}$. The basic equation in the new hydrodynamics for the normal component of the velocity is the equation of motion (4.1), which coincides with the motion equation of a non-relativistic particle for which we choose a system of units such that the speed of light, the charge of the particle and its mass, equal unity.

The new terms on the right-hand side of (4.1) have a simple mechanical interpretation. The Lorenz force $[\boldsymbol{v}_n \times \boldsymbol{H}]$ is simply but the Coriolis force. The addition in φ of the pressure p, equal to $v_\tau{}^2/2$, has a direct connection to the Bernoulli formula. The term $\partial \boldsymbol{v}_\tau/\partial t$ in the expression for the electric field $\boldsymbol{E}$ appears due to the transition to a movable, non-inertial system of coordinates.

The equation of motion (4.1) is written in the Eulerian representation. To obtain its Lagrangian formulation, one needs to consider the equations for new 'trajectories' determined by the velocity $\boldsymbol{v}_n$ (compare with (3.8)):

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}_n(\boldsymbol{r}, t) \tag{4.4}$$

with the initial conditions

$$\boldsymbol{r}|_{t=0} = \boldsymbol{a}.$$

The solution of equation (4.4) yields the mapping

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{a}, t), \tag{4.5}$$

which defines a transition from the Eulerian description to the new Lagrangian one.

The equations of motion in the new variables have the canonical Hamiltonian form

$$\dot{\boldsymbol{P}} = -\frac{\partial h}{\partial \boldsymbol{r}}, \qquad \dot{\boldsymbol{r}} = \frac{\partial h}{\partial \boldsymbol{P}}, \tag{4.6}$$

where, as before, the overdots denote differentiation with respect to time for fixed $\boldsymbol{a}$, $\boldsymbol{P} = \boldsymbol{v}_n + \boldsymbol{A} \equiv \boldsymbol{v}$ is the generalized momentum, and the Hamiltonian of a particle h (being a function of the momentum $\boldsymbol{P}$ and coordinate $\boldsymbol{r}$) is given by the standard expression for a charged particle moving in an electromagnetic field (see e.g. Landau & Lifshitz 1980):

$$h = \frac{1}{2}(\boldsymbol{P} - \boldsymbol{A})^2 + \varphi \equiv p + \frac{\boldsymbol{v}^2}{2},$$

i.e. it coincides with the Bernoulli 'invariant'.

The first equation in (4.6) is the equation of motion (4.1), which is written in terms of $\boldsymbol{a}$ and t, and the second equation coincides with (4.4).

For the new hydrodynamics (4.1) and its Hamiltonian version (4.6) it is possible to formulate a 'new' Kelvin theorem (it is also the Liouville theorem)

$$\Gamma = \oint (\boldsymbol{P} \cdot d\boldsymbol{r}),$$

where the integration is performed along a loop moving together with the 'fluid'. Hence, analogously to the derivation of (2.3), we obtain the expression for a new Cauchy invariant:

$$\boldsymbol{I} = \mathrm{curl}_a \left(P_i \frac{\partial x_i}{\partial \boldsymbol{a}} \right). \tag{4.7}$$

Its difference from the original Cauchy invariant is in the definition of trajectories. In (4.7), trajectories are given by the normal velocity component $\boldsymbol{v}_n$ instead of the full velocity $\boldsymbol{v}$ in (2.3). As a consequence, the 'new' hydrodynamics becomes compressible: $\mathrm{div}\, \boldsymbol{v}_n \neq 0$. Therefore there are no restrictions imposed on the Jacobian J of the mapping (4.5). The Jacobian J can thus take arbitrary values.

From formula (4.7) it is easy to find an expression for the vorticity $\boldsymbol{\Omega}$ at a given point $\boldsymbol{r}$ at any instant t (compare with Kuznetsov & Ruban 1998, 2000*b*):

$$\boldsymbol{\Omega}(\boldsymbol{r}, t) = \frac{(\boldsymbol{\Omega}_0(\boldsymbol{a}) \cdot \nabla_a)\boldsymbol{r}(a, t)}{J}, \tag{4.8}$$

where J is the Jacobian of the mapping (4.4),

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)}.$$

Here we took into account that the generalized momentum $\boldsymbol{P}$ coincides with the velocity $\boldsymbol{v}$, including the initial moment of time $t = 0$: $\boldsymbol{P}_0(\boldsymbol{a}) \equiv \boldsymbol{v}_0(\boldsymbol{a})$. The vorticity $\boldsymbol{\Omega}_0(\boldsymbol{a})$ in this relation is the 'new' Cauchy invariant with zero divergence, $\mathrm{div}_a \boldsymbol{\Omega}_0(a) = 0$.

Equation (4.8) is the central equation of the vortex line representation. It generalizes the relation (3.4) to arbitrary topology of the vortex lines. The variables $\boldsymbol{a}$ in this expression can be considered locally as a set of λ, μ, and s.

4.1. *Weber-type transform*

As we saw in § 2, the Cauchy invariant can be obtained from the Weber transformation. The same statement is also valid for the new hydrodynamics of a charged fluid.

Consider the one-form $\omega = (\boldsymbol{P} \cdot \mathrm{d}\boldsymbol{r})$ and calculate its time derivative. From the equations of motion (4.6) we obtain

$$\dot{\omega} = \mathrm{d}[-h + (\boldsymbol{P} \cdot \dot{\boldsymbol{r}})],$$

where the dot has the same meaning as in (4.6). Hence it follows that the vector function

$$u_k = \frac{\partial x_i}{\partial a_k} P_i,$$

which depends on t and $\boldsymbol{a}$, will obey the following equation of motion (compare with (2.5)):

$$\dot{u}_k = \frac{\partial}{\partial a_k} \left(-p + \frac{v_n^2}{2} - \frac{v_\tau^2}{2} \right).$$

Integration of this equation with respect to time gives the Weber-type transformation:

$$u_k(\boldsymbol{a}, t) = u_{k0}(\boldsymbol{a}) + \frac{\partial \Phi}{\partial a_k}, \tag{4.9}$$

where the potential Φ satisfies the non-stationary Bernoulli equation:

$$\dot{\Phi} = -p + \frac{v_n^2}{2} - \frac{v_\tau^2}{2}.$$

If $\Phi|_{t=0} = 0$, then the time-independent vector $\boldsymbol{u}_0(\boldsymbol{a})$ coincides with the initial velocity $\boldsymbol{v}_0(\boldsymbol{a})$. Applying the curl operator to the relation (4.9), we again obtain the Cauchy invariant (4.7).

Thus, in the general situation the equation of motion for the vortex lines has the form of (4.4), which is complemented by the relation (4.8) and equation

$$\boldsymbol{\Omega}(\boldsymbol{r}, t) = \mathrm{curl}_r \boldsymbol{v}(\boldsymbol{r}, t) \tag{4.10}$$

with the additional constraint of $\mathrm{div}_r \boldsymbol{v}(\boldsymbol{r}, t) = 0$. All these equations comprise the *vortex line representation* of the Euler equations.

Another important property of the vortex line representation is the absence of any restrictions on the value of the Jacobian J. Such restrictions do exist, for instance, for the transition from the Eulerian to the Lagrangian description in the original Euler equation (2.1), where the Jacobian in the simplest situation equals unity. The quantity $1/J$ for the system (4.4), (4.10), (4.8) represents the density n of vortex lines. This quantity, as a function of $\boldsymbol{r}$ and t, according to (4.4), obeys the continuity equation

$$\frac{\partial n}{\partial t} + \mathrm{div}_r (n \boldsymbol{v}_n) = 0.$$

In this equation $\mathrm{div}_r \boldsymbol{v}_n \neq 0$ because only the total velocity has vanishing divergence. For some symmetrical flows the velocity $\boldsymbol{v}$ can be transverse to the vorticity field, for instance for two-dimensional flows or axially symmetric flows without swirl. The Jacobian for all such flows remains fixed and cannot change in time.

Equations of motion (4.4) and (4.10), together with the relation (4.8), can be considered as the result of partial integration of the Euler equations (2.1). These new equations are resolved with respect to the Cauchy invariants – an infinite number of integrals of motion; this is a very important issue for numerical integration (see Zheligovsky *et al.* 2001; Kuznetsov *et al.* 2003). For the partially integrated system, the Cauchy invariants are conserved automatically; however, during the numerical integration of the Euler equations one needs to test their conservation. These invariants could be useful to determine the accuracy of discrete algorithms for the direct integration of the Euler equations.

The vortex line representation, (4.10), (4.8) and (4.4), has advantages and disadvantages from the point of view of simulation of the Euler equations. First, in these equations the temporal and spatial derivatives are separated from each other which is an evident advantage. Second, for a periodic domain the velocity $\boldsymbol{v}$ can be easily found by inverting the curl operator in the physical $\boldsymbol{r}$-space using the standard FFT. However, to integrate equation (4.4) one needs to know how the normal velocity depends on $\boldsymbol{a}$. This makes it necessary to consider two grids corresponding to the $\boldsymbol{r}$-space and auxiliary $\boldsymbol{a}$-space, and is a drawback of this description.

We now present two formulations of equations (4.8), (4.10), and (4.4) in which all the quantities depend on either $\boldsymbol{r}$ and t or on $\boldsymbol{a}$ and t.

In the first case let us consider the mapping inverse to (4.5), $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{r}, t)$. It is evident that $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{r}, t)$ will obey the equation

$$\frac{\partial \boldsymbol{a}}{\partial t} + (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{a} = 0. \tag{4.11}$$

In the $(\boldsymbol{r}, t)$-variables, formula (4.8) takes the form

$$\boldsymbol{\Omega}(\boldsymbol{r}, t) = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}\Omega_{0\alpha}(\boldsymbol{a})[\nabla a_\beta \times \nabla a_\gamma], \tag{4.12}$$

where $\Omega_0(\boldsymbol{a})$ is the initial vorticity. These two equations together with (4.10) under the condition $\mathrm{div}_r \boldsymbol{v} = 0$ represent a closed system of the VLR equations written in the $\boldsymbol{r}$-space.

To write (4.4), (4.8) and (4.10) in the $(\boldsymbol{a}, t)$-variables, one needs to transform (4.10) and the incompressibility condition $\nabla_r \cdot \boldsymbol{v} = 0$. Instead of (4.10) we have equation (4.7)

$$\Omega_0(\boldsymbol{a}) = \mathrm{curl}_a \left(v_i \frac{\partial x_i}{\partial \boldsymbol{a}} \right). \tag{4.13}$$

The incompressibility condition in these variables transforms into the equation

$$\varepsilon_{ijk}(\nabla_a v_i \cdot [\nabla_a x_j \times \nabla_a x_k]) = 0, \tag{4.14}$$

where we have used the relation

$$\frac{\partial a_\alpha}{\partial x_i} = \frac{1}{2J}\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}\frac{\partial x_j}{\partial a_\beta}\frac{\partial x_k}{\partial a_\gamma}. \tag{4.15}$$

Equations (4.4), (4.13), together with the relation (4.14) form the complete VLR system in variables $\boldsymbol{a}$ and t. In this case equation (4.8) serves as the definition of vorticity in the physical space.

4.2. *Topological constraints of the VLR*

The vortex line representation as a local change of variables $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{a}, t)$ is invalid at singular points where vorticity equals zero and hence the normal velocity is undetermined. Due to the 'frozenness' of the vorticity such points remain, being advected by the fluid. Let us consider the point $\boldsymbol{r} = \boldsymbol{r}(t)$ that is defined by equation $\boldsymbol{\Omega}(\boldsymbol{r}(t), t) = 0$. Upon differentiating this equation with respect to time, we obtain

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + (\dot{\boldsymbol{r}}(t) \cdot \nabla)\boldsymbol{\Omega} = 0,$$

which coincides with the Euler equation for vorticity (1.1) in this partial case, $\boldsymbol{\Omega}(\boldsymbol{r}(t), t) = 0$; here $\dot{\boldsymbol{r}}(t) = \boldsymbol{v}(\boldsymbol{r}(t), t)$. This proves that these points are advected by flows and cannot dissipate or, for instance, transform into cuts.

The velocity $\boldsymbol{v}$ at these points can be determined by inverting the curl operator: $\boldsymbol{v} = \mathrm{curl}^{-1}\boldsymbol{\Omega}$. However, the normal component of the velocity $\boldsymbol{v}_n$ is not defined at these points. For this reason, the null points of the vector field $\boldsymbol{\tau}(\boldsymbol{r}) \equiv \boldsymbol{\Omega}/|\Omega|$ (i.e. the unit tangent vector to the vortex lines) represent topological singularities which can be classified using topological methods. This classification is determined by the topological charge as the degree of the mapping $\mathscr{S}^2 \to \mathscr{S}^2$, given by the integral (see e.g. Volovik & Mineev 1977)

$$\int_{\partial V} \epsilon_{\alpha\beta\gamma}(\tau \cdot [\partial_\beta \tau \times \partial_\gamma \tau])\, \mathrm{d}S_\gamma = 4\pi m, \tag{4.16}$$

where the integration is taken over the boundary ∂V of the region V containing the points, and the topological charge m has integer values.

Thus, equations (4.8) and (4.4) together with the condition (4.16) constitute a complete system of equations that provides a vortex line representation for the Euler equations in the general case.

It is natural to call this new charged hydrodynamics the hydrodynamics of vortex lines. The main property of the new hydrodynamics is *compressibility* of the new 'fluid'. The VLR as a mapping is also compressible. Its Jacobian does not equal unity; for instance, it can vanish. As known from gasdynamics, the process of the Jacobian vanishing is nothing more than breaking. In the present case this is the breaking of vortex lines that, according to (4.8), should result in infinite vorticity. Numerical evidence of breaking in ideal fluids (Zheligovsky *et al.* 2001; Kuznetsov *et al.* 2003), as well as the existence of breaking of vortex lines for three-dimensional integrable hydrodynamics (Kuznetsov & Ruban 2000*a*), is in favour of such a mechanism. The latter model is also incompressible, and has the same symplectic operator for the Poisson brackets as that for the Euler equations but differs from the Euler equations by the form of the Hamiltonian $H = \int |\boldsymbol{\Omega}| \mathrm{d}\boldsymbol{r}$, where $\boldsymbol{\Omega}$ is understood as a generalized vorticity. This model can be obtained from the Euler equations in the so-called local induction approximation (for the details of the approximation see Ricca (1991) and references therein). However, for symmetric flows the Jacobian can be fixed and consequently breaking of vortex lines will be forbidden. For instance, this is the case for two-dimensional flows as well as for axisymmetric flows without swirl when the velocity is normal to the vorticity lines. The latter is in complete correspondence with the proof given by Majda (1986) that in the case of axisymmetric flows without swirl the solution of the Euler equations remains smooth for any time $t > 0$.

5. VLR for viscous fluids

Consider now the question about the application of the VLR to flows of viscous fluids described by the Navier–Stokes equations. For the vorticity $\boldsymbol{\Omega}$ these equations are written in the form

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - \mathrm{curl}[\boldsymbol{v} \times \boldsymbol{\Omega}] = \nu \mathrm{curl}\mathrm{curl}\, \boldsymbol{\Omega}. \tag{5.1}$$

If in these equations we transform the variables $\boldsymbol{r}$ into the new variables $\boldsymbol{a}$ by using (4.5) and, in particular, the vorticity $\boldsymbol{\Omega}$ into $\boldsymbol{\Omega}_0$ using the Cauchy relation (4.8), where $\boldsymbol{\Omega}_0$ is assumed to depend not only on $\boldsymbol{a}$ but also on the time t, $\boldsymbol{\Omega}_0 = \boldsymbol{\Omega}_0(\boldsymbol{a}, t)$, then all the terms on the left-hand side of (5.1) cancel out due to the equation (4.4) together with the Cauchy relation (4.8), except the derivative of $\boldsymbol{\Omega}_0$ with respect to t,

$$\frac{1}{J}\left(\frac{\partial \boldsymbol{\Omega}_0}{\partial t} \cdot \nabla_a\right) \boldsymbol{r} = -\nu \mathrm{curl}\mathrm{curl} \boldsymbol{\Omega}. \tag{5.2}$$

Here all the quantities on the left-hand side are functions of $\boldsymbol{a}$ and t, but the right-hand side is written in the Eulerian variables $\boldsymbol{r}$ and t. Therefore to obtain an equation for $\boldsymbol{\Omega}_0(\boldsymbol{a}, t)$, we need to express the left-hand side of (5.2) in terms of $\boldsymbol{a}$ and t, and then to invert the Jacobi matrix $\hat{J}$.

After inverting the Jacobi matrix $\hat{J}$ we can resolve the equation with respect to the time-derivative of $\boldsymbol{\Omega}_0$,

$$\frac{\partial \Omega_{0\alpha}}{\partial t} = -\nu J \frac{\partial a_\alpha}{\partial x_i} \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left\{\varepsilon_{klm} \frac{\partial}{\partial x_l}\left[\frac{1}{J} \frac{\partial x_m}{\partial a_\beta} \Omega_{0\beta}\right]\right\},$$

where ε_{ijk} is the absolutely antisymmetric tensor with $\varepsilon_{123} = 1$. In the next step we rewrite the derivatives with respect to $\boldsymbol{r}$ in terms of the derivatives with respect to $\boldsymbol{a}$, which gives

$$\frac{\partial \Omega_{0\alpha}}{\partial t} = -\nu J\, \varepsilon_{ijk} \frac{\partial a_\alpha}{\partial x_i} \frac{\partial a_\beta}{\partial x_j} \frac{\partial}{\partial a_\beta}\left\{\varepsilon_{klm} \frac{\partial a_\gamma}{\partial x_l} \frac{\partial}{\partial a_\gamma}\left[\frac{1}{J} \frac{\partial x_m}{\partial a_\beta} \Omega_{0\beta}\right]\right\}. \tag{5.3}$$

For the further transformation we use two relations between components of the Jacobi matrix $\hat{J}$ and its inverse $\hat{J}^{-1}$:

$$J\,\varepsilon_{ijk}\,\frac{\partial a_\alpha}{\partial x_i}\,\frac{\partial a_\beta}{\partial x_j}=\varepsilon_{\alpha\beta\gamma}\,\frac{\partial x_k}{\partial a_\gamma},\qquad \varepsilon_{kml}\,\frac{\partial a_\gamma}{\partial x_l}=\frac{1}{J}\,\varepsilon_{\gamma\kappa\mu}\,\frac{\partial x_k}{\partial a_\kappa}\,\frac{\partial x_m}{\partial a_\mu}.$$

Note that both these relations are simple sequences of the formula (4.15). Applying these relations to (5.3) yields

$$\frac{\partial\Omega_{0\alpha}}{\partial t}=\nu\varepsilon_{\alpha\beta\gamma}\frac{\partial x_k}{\partial a_\gamma}\frac{\partial}{\partial a_\beta}\left\{\frac{1}{J}\,\varepsilon_{\nu\kappa\mu}\frac{\partial x_k}{\partial a_\kappa}\frac{\partial x_m}{\partial a_\mu}\frac{\partial}{\partial a_\nu}\left[\frac{1}{J}\,\frac{\partial x_m}{\partial a_\delta}\,\Omega_{0\delta}\right]\right\}.$$

In this equation, due to the antisymmetry of the tensor $\varepsilon_{\gamma\kappa\mu}$ we can bring the derivatives $\partial x_k/\partial a_\gamma$ and $\partial x_m/\partial a_\mu$ into the corresponding brackets, which results in

$$\frac{\partial\Omega_{0\alpha}}{\partial t}=-\nu\varepsilon_{\alpha\beta\gamma}\frac{\partial}{\partial a_\beta}\left\{\frac{1}{J}\,g_{\gamma\kappa}\varepsilon_{\kappa\nu\mu}\frac{\partial}{\partial a_\nu}\left[\frac{1}{J}\,g_{\mu\delta}\Omega_{0\delta}\right]\right\},$$

where we introduced the metric tensor

$$g_{\alpha\beta}=\frac{\partial x_i}{\partial a_\alpha}\cdot\frac{\partial x_i}{\partial a_\beta}.$$

In the vector notation this equation takes the more compact form

$$\frac{\partial\boldsymbol{\Omega}_0}{\partial t}=-\nu\,\mathrm{curl}_a\left(\frac{\hat{g}}{J}\,\mathrm{curl}_a\left(\frac{\hat{g}}{J}\boldsymbol{\Omega}_0\right)\right). \tag{5.4}$$

Formally this is a linear equation for $\boldsymbol{\Omega}_0$, but it contains geometric characteristics of the VLR expressed only in terms of the metric tensor $\hat{g}$ (the Jacobian $J=\sqrt{\det\hat{g}}$).

Equation (5.4) for the Cauchy invariant formally coincides with that obtained by Yakubovich & Zenkovich (2002) for incompressible hydrodynamics, where the variables $\boldsymbol{a}$ are assumed to be the Lagrangian markers of the fluid particles. In the Zenkovich–Yakubovich equation the Jacobian J is proposed to be independent of time, and in the simplest case equal to unity. This is the principal difference between the Zenkovich–Yakubovich equation and (5.4). The Jacobian J in (5.4) is a function of the time t and coordinates $\boldsymbol{a}$.

A remarkable peculiarity of the system obtained is the *exact separation* of two different temporal scales responsible for the inertial (in fact, nonlinear) and viscous processes, respectively. The former are described by equation (4.4), and the latter by the diffusive-type equation (5.4), where the diffusion 'coefficient', which is proportional to the viscosity ν, is defined by the metric of the mapping $\boldsymbol{r}=\boldsymbol{r}(\boldsymbol{a},t)$.

6. Conclusions

A vortex line representation has been introduced to describe vortical flows for both ideal and viscous fluids. The vorticity changes in ideal fluids due to the *normal* velocity component only; the velocity component parallel to the vorticity field plays a passive role, providing the incompressibility of the flow. Therefore the divergence of the normal velocity does not vanish. This is the cause of the VLR being a compressible mapping. The VLR for ideal fluids locally corresponds to the mixed Lagrangian–Eulerian description, in which each vortex line is labelled by two Lagrangian markers (the Clebsch variables may be used as such markers) and parameterized by an Eulerian coordinate. The transition to the VLR in the Euler equations leads to a new *compressible* hydrodynamics of a charged fluid in a self-consistent electromagnetic field. This new hydrodynamics can be called the hydrodynamics of vortex lines when the normal velocity component plays the role of the velocity of the vortex lines. The compressible character of the VLR can be a reason for the appearance of infinities in the vorticity that can be interpreted as the result of the vortex line breaking when one vortex line touches another vortex line. Near the breaking point the parallel velocity component must not have zero values. This is one of the main indicators for such type of processes.

It has also been shown that the VLR can be introduced as well for the description of ideal fluid flows with a non-trivial topology when the vorticity has null points, i.e. topological singularities that persist in time. In terms of the VLR the three-dimensional Euler equations become resolved with respect to an infinite number of integrals (the Cauchy invariants) which is an important issue for numerical solution of the Euler equations. On the contrary, when performing a direct numerical integration of the Euler equations, one needs to check the conservation of these invariants.

Finally, it has been shown that the application of the VLR to vortical viscous flows leads to an exact splitting of the equations into two systems that can be interpreted as an exact separation of the inertial (in fact, the nonlinear) from the viscous process.

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